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A SHOCK-CAPTURING SCHEME FOR BODY-FITTED MESHES

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SUMMARY

A finite difference scheme based on flux difference splitting is presented for the solution of the two-dimensional Euler equations of gas dynamics in a generalized co-ordinate system. The scheme is based on numerical characteristic decomposition and solves locally linearized Riemann problems using upwind differencing. The decomposition is for a generalized co-ordinate system and a convex equation of state. This ensures good shock-capturing properties when incorporated with operator splitting and the advantage of using body-fitted co-ordinates. The resulting scheme is applied to supersonic flow of 'real air' past a circular cylinder.

KEY WORDS Euler Equations Riemann Solver Generalized Co-ordinates

1. INTRODUCTION

The approximate (linearized) Riemann solver of Roe[1] has proved to be successful in its application to the compressible flow of an ideal gas in one dimension,[2] and in two dimensions when incorporated with operator splitting.[3] A similar Riemann solver was proposed by Glaister[4] for compressible flows in one dimension and general convex equations of state. This scheme was extended[5] to the two-dimensional Euler equations in Cartesian co-ordinates and general convex equations of state using operator splitting.

In this paper we seek to extend further the analysis of Glaister to a generalized co-ordinate system. The resulting scheme can be used with non-Cartesian, body-fitted meshes in two dimensions. The area of generating body-fitted meshes is one that is increasing in importance.[6,7]

In Section 2 we consider the Jacobian matrix of one of the flux functions for the Euler equations in a generalized co-ordinate system, and in Section 3 derive an approximate Riemann solver for the solution of these equations. Finally, in Section 4 we describe a two-dimensional test problem and display the numerical results achieved using the scheme of Section 3.

2. EULER EQUATIONS

In this section we state the equations of motion for an inviscid, compressible fluid in two dimensions in terms of two generalized space co-ordinates. We also give the eigenvalues and eigenvectors of the Jacobian of one of the corresponding flux functions.

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2.1. Equations of flow

The two-dimensional Euler equations for the flow of an inviscid, compressible fluid can be written in generalized co-ordinates ξ, η as

$$(J\mathbf{w})_t + \mathbf{F}_\xi + \mathbf{G}_\eta = \mathbf{0}, \tag{1}$$

where

$$\mathbf{w} = (\rho, \rho u, \rho v, e)^T, \tag{2}$$

$$\mathbf{F}(\mathbf{w}) = (\rho U, y_\eta p + \rho u U, -x_\eta p + \rho v U, U(e+p))^T, \tag{3}$$

$$\mathbf{G}(\mathbf{w}) = (\rho V, -y_\xi p + \rho u V, x_\xi p + \rho v V, V(e+p))^T, \tag{4}$$

$$e = \rho i + \tfrac{1}{2}\rho(u^2 + v^2) \tag{5}$$

and

$$U = y_\eta u - x_\eta v, \qquad V = x_\xi v - y_\xi u. \tag{6a, b}$$

The Jacobian of the grid transformation $x = x(\xi, \eta)$, $y = y(\xi, \eta)$ from Cartesian co-ordinates x, y to generalized co-ordinates ξ, η is given by

$$J = x_\xi y_\eta - x_\eta y_\xi. \tag{7}$$

The quantities $\rho = \rho(\xi, \eta, t)$, $u = u(\xi, \eta, t)$, $v = v(\xi, \eta, t)$, $p = p(\xi, \eta, t)$, $i = i(\xi, \eta, t)$ and $e = e(\xi, \eta, t)$ represent the density, the velocity in the x and y co-ordinate directions, the pressure, the specific internal energy and the total energy respectively at a general position ξ, η in space and at time t. In addition, we have an equation of state of the form

$$p = p(\rho, i). \tag{8}$$

In the case of an ideal gas, equation (8) takes the form

$$p = (\gamma - 1)\rho i, \tag{9}$$

where γ is the ratio of specific heat capacities of the fluid.

2.2. Structure of the Jacobian

We now give the Jacobian of the flux function $\mathbf{F}(\mathbf{w})$, and its eigenvalues and eigenvectors, since this information, together with similar information for the Jacobian of $\mathbf{G}(\mathbf{w})$, will form the basis for the approximate Riemann solver.

The Jacobian $\mathbf{A} = \partial\mathbf{F}/\partial\mathbf{w}$ of the flux function $\mathbf{F}(\mathbf{w})$ is given by

$$\mathbf{A} = \begin{bmatrix} 0 & y_\eta & -x_\eta & 0 \\ y_\eta\left(a^2 - \frac{p_i}{\rho}(H - q^2)\right) - uU & U + y_\eta u\left(1 - \frac{p_i}{\rho}\right) & -x_\eta u - y_\eta v\frac{p_i}{\rho} & y_\eta\frac{p_i}{\rho} \\ -x_\eta\left(a^2 - \frac{p_i}{\rho}(H - q^2)\right) - uU & y_\eta v + x_\eta u\frac{p_i}{\rho} & U - x_\eta v\left(1 - \frac{p_i}{\rho}\right) & -x_\eta\frac{p_i}{\rho} \\ U\left(a^2 - H - \frac{p_i}{\rho}(H - q^2)\right) & y_\eta H - uU\frac{p_i}{\rho} & -x_\eta H - vU\frac{p_i}{\rho} & U\left(1 + \frac{p_i}{\rho}\right) \end{bmatrix}. \tag{10}$$

where the fluid speed q, enthalpy H and sound speed a are given by

$$q^2 = u^2 + v^2, \tag{11}$$

$$H = p/\rho + i + \tfrac{1}{2}q^2 \tag{12}$$

and

$$a^2 = pp_i/\rho^2 + p_\rho, \tag{13}$$

and the quantities p_i, p_ρ denote the derivatives $\partial p(\rho, i)/\partial i|_\rho$, $\partial p(\rho, i)/\partial \rho|_i$ respectively. The eigenvalues λ_i of $\mathbf{A}$ are given by

$$\lambda_{1,2,3,4} = U \pm a\sqrt{(x_\eta^2 + y_\eta^2)}, \quad U, U \tag{14a–d}$$

with corresponding eigenvectors

$$\mathbf{e}_{1,2} = \left(1, u \pm \frac{ay_\eta}{\sqrt{(x_\eta^2 + y_\eta^2)}}, v \mp \frac{ax_\eta}{\sqrt{(x_\eta^2 + y_\eta^2)}}, H \pm \frac{aU}{\sqrt{(x_\eta^2 + y_\eta^2)}}\right)^{\mathrm{T}}, \tag{15a, b}$$

$$\mathbf{e}_3 = \left(1, u, v, i + \tfrac{1}{2}(u^2 + v^2) - \frac{\rho p_\rho}{p_i}\right)^{\mathrm{T}} \tag{15c}$$

and

$$\mathbf{e}_4 = (0, x_\eta, y_\eta, x_\eta u + y_\eta v)^{\mathrm{T}}. \tag{15d}$$

Similar results hold for the Jacobian of $\mathbf{G}(\mathbf{w})$ as follows. The eigenvalues of $\mathbf{G}$ are

$$\lambda_{1,2,3,4} = V \pm a\sqrt{(x_\xi^2 + y_\xi^2)}, \; V, V$$

with corresponding eigenvectors

$$\mathbf{e}_{1,2} = \left(1, u \mp \frac{ay_\xi}{\sqrt{(x_\xi^2 + y_\xi^2)}}, v \pm \frac{ax_\xi}{\sqrt{(x_\xi^2 + y_\xi^2)}}, H \pm \frac{aU}{\sqrt{(x_\xi^2 + y_\xi^2)}}\right)^{\mathrm{T}},$$

$$\mathbf{e}_3 = (1, u, v, i + \tfrac{1}{2}(u^2 + v^2) - \rho p_\rho / p_i)^{\mathrm{T}}$$

and

$$\mathbf{e}_4 = (0, -x_\xi, -y_\xi, -x_\xi u - y_\xi v)^{\mathrm{T}}.$$

In the next section we develop an approximate Riemann solver based on the results of this section.

3. APPROXIMATE RIEMANN SOLVER

We propose solving equations (1)–(8) using operator splitting, i.e. we solve successively

$$\tfrac{1}{2}(J\mathbf{w})_t + \mathbf{F}_\xi = \mathbf{0} \tag{16a}$$

and

$$\tfrac{1}{2}(J\mathbf{w})_t + \mathbf{G}_\eta = \mathbf{0} \tag{16b}$$

along ξ and η co-ordinate lines respectively. We describe the scheme for solving equation (16a), and the solution of equation (16b) will follow in a similar way.

3.1. Linearized Riemann problem

If the solution of equation (16a) is sought along a ξ co-ordinate line given by $\eta = \eta_0$, a constant, using a finite difference method, then the solution is known at a set of discrete mesh points

$(\xi, \eta, t)=(\xi_j, \eta_0, t_n)$ at any time t_n. Following Godunov,[8] the approximate solution $\mathbf{w}_j^n$ to $\mathbf{w}$ at (ξ_j, η_0, t_n) can be considered as a set of piecewise constants $\mathbf{w}=\mathbf{w}_j^n$ for $\xi\in(\xi_j-\Delta\xi/2, \xi_j+\Delta\xi/2)$ at time t_n, where $\Delta\xi=\xi_j-\xi_{j-1}$ is a constant mesh spacing. A Riemann problem is now present at each interface $\xi_{j-1/2}=\frac{1}{2}(\xi_{j-1}+\xi_j)$ separating adjacent states $\mathbf{w}_{j-1}^n$, $\mathbf{w}_j^n$. We consider solving the linearized Riemann problem

$$\tfrac{1}{2}(J\mathbf{w})_t+\tilde{\mathbf{A}}(\mathbf{w}_{j-1}^n, \mathbf{w}_j^n)\mathbf{w}_\xi=\mathbf{0}, \tag{17}$$

where $\tilde{\mathbf{A}}_{j-1/2}=\tilde{\mathbf{A}}(\mathbf{w}_{j-1}^n, \mathbf{w}_j^n)$ is an approximation to the Jacobian $\mathbf{A}$ and is a constant matrix depending on the states either side of $\xi_{j-1/2}$. This matrix $\tilde{\mathbf{A}}_{j-1/2}$ will be required to satisfy the following three properties:

(i) $\tilde{\mathbf{A}}_{j-1/2}(\mathbf{w}_{j-1}^n, \mathbf{w}_j^n)\to\mathbf{A}(\mathbf{w})$ as $\mathbf{w}_{j-1}^n\to\mathbf{w}_j^n\to\mathbf{w}$.
(ii) $\tilde{\mathbf{A}}_{j-1/2}$ has four linearly independent eigenvectors.
(iii) $\Delta\mathbf{F}=\tilde{\mathbf{A}}_{j-1/2}\Delta\mathbf{w}$.

These properties were shown by Roe[1] in the ideal gas case in Cartesian co-ordinates to guarantee conservation and have good one-dimensional shock-capturing properties.

3.2. Numerical scheme

Once such a matrix has been constructed, equation (17) can be solved approximately as

$$J_{j-1/2}\frac{\mathbf{w}_k^{n+1}-\mathbf{w}_k^n}{2\Delta t}+\tilde{\mathbf{A}}_{j-1/2}\frac{\mathbf{w}_j^n-\mathbf{w}_{j-1}^n}{\Delta\xi}=\mathbf{0}, \tag{18}$$

where k can be $j-1$ or j, $\Delta t=t_{n+1}-t_n$ is a constant time step and $J_{j-1/2}$ is an approximation to the grid Jacobian at $(\xi, \eta)=(\xi_{j-1/2}, \eta_0)$. If we project

$$\Delta\mathbf{w}=\mathbf{w}_j^n-\mathbf{w}_{j-1}^n=\sum_{i=1}^{4}\tilde{\alpha}_i\tilde{\mathbf{e}}_i, \tag{19}$$

where $\tilde{\mathbf{e}}_i$ are the eigenvectors of $\tilde{\mathbf{A}}_{j-1/2}$, then equation (18) can be written as

$$J_{j-1/2}\frac{\mathbf{w}_k^{n+1}-\mathbf{w}_k^n}{2\Delta t}+\frac{\sum_{i=1}^{4}\tilde{\lambda}_i\tilde{\alpha}_i\tilde{\mathbf{e}}_i}{\Delta\xi}=\mathbf{0}, \tag{20}$$

where $\tilde{\lambda}_i$ are the eigenvalues of $\tilde{\mathbf{A}}_{j-1/2}$. Equation (20) now gives rise to the following first-order upwind algorithm:

$$\mathbf{w}_{j-1}^{n+1}=\mathbf{w}_{j-1}^n-\frac{2\Delta t}{J_{j-1/2}\Delta\xi}\tilde{\lambda}_i\tilde{\alpha}_i\tilde{\mathbf{e}}_i \quad \text{if } \tilde{\lambda}_i<0 \tag{21a}$$

and

$$\mathbf{w}_j^{n+1}=\mathbf{w}_j^n-\frac{2\Delta t}{J_{j-1/2}\Delta\xi}\tilde{\lambda}_i\tilde{\alpha}_i\tilde{\mathbf{e}}_i \quad \text{if } \tilde{\lambda}_i>0. \tag{21b}$$

This is a form of the scheme written in a cell-based manner. Equations (21a, b) represent the increment of $\mathbf{w}_{j-1}^n$, $\mathbf{w}_j^n$ due to the interface at $\xi_{j-1/2}$. There will be similar increments to $\mathbf{w}_{j-1}^n$ due to the interface at $\xi_{j-3/2}$ when considering the cell (ξ_{j-2}, ξ_{j-1}), and to $\mathbf{w}_j^n$ due to the interface at $\xi_{j+1/2}$ when considering the cell (ξ_j, ξ_{j+1}).

Extensions of this first-order algorithm to second order[9] and to non-uniform grids[10,11] can be made.

3.3. Grid generation and grid Jacobian

The purpose of this paper is to present an approximate Riemann solver for use with non-Cartesian body-fitted co-ordinates. The mapping from physical $(x-y)$ space to computational $(\xi-\eta)$ space can be given analytically or constructed numerically.[12] In the case where the mapping $x=x(\xi,\eta)$, $y=y(\xi,\eta)$ is known analytically, we can approximate $J_{j-1/2}$ in equation (18) as

$$J_{j-1/2}=(x_\xi y_\eta - x_\eta y_\xi)(\xi_{j-1/2},\eta_0); \tag{21}$$

alternatively, $J_{j-1/2}$ can be approximated using central differences. In addition, we will need a suitable approximation $x_\eta^{j-1/2}$ for x_η at $(\xi_{j-1/2},\eta_0)$, and in the analytic case we take

$$x_\eta^{j-1/2}=x_\eta(\xi_{j-1/2},\eta_0) \tag{22}$$

as in equation (21); otherwise we set $x_\eta^{j-1/2}$ to be the arithmetic mean of central difference approximations to x_η at (ξ_{j-1},η_0) and (ξ_j,η_0). Similar approximations hold for x_ξ, y_η and y_ξ.

3.4. Construction of $\tilde{\mathbf{A}}_{j-1/2}$

Consider a ξ co-ordinate line given by $\eta=\eta_0$, a constant, and denote points ξ_{j-1}, ξ_j on this line by ξ_L, ξ_R respectively. In addition, we denote $\mathbf{w}_{j-1}^n=\mathbf{w}_L$, $\mathbf{w}_j^n=\mathbf{w}_R$ and assume that $X=x_\eta^{j-1/2}$, $Y=y_\eta^{j-1/2}$ denote approximations to x_η, y_η that are constant in the interval (ξ_L,ξ_R). Our aim is to construct a matrix $\tilde{\mathbf{A}}_{j-1/2}=\tilde{\mathbf{A}}(\mathbf{w}_L,\mathbf{w}_R)$ satisfying properties (i)–(iii) of Section 3.2. Equivalently, we could find average eigenvalues $\tilde{\lambda}_i$ and average eigenvectors $\tilde{\mathbf{e}}_i$ of the Jacobian $\mathbf{A}$ at ξ_L, ξ_R given by equations (14a)–(15d) such that

$$\Delta\mathbf{w}=\sum_{i=1}^{4}\tilde{\alpha}_i\tilde{\mathbf{e}}_i \tag{23a–d}$$

and

$$\Delta\mathbf{F}=\sum_{i=1}^{4}\tilde{\lambda}_i\tilde{\alpha}_i\tilde{\mathbf{e}}_i \tag{24a–d}$$

for some wavestrengths $\tilde{\alpha}_i$, where

$$\Delta(\cdot)=(\cdot)_R-(\cdot)_L. \tag{25}$$

This yields the approximate Jacobian

$$\tilde{\mathbf{A}}_{j-1/2}=\tilde{\mathbf{M}}_{j-1/2}\tilde{\mathbf{D}}_{j-1/2}\tilde{\mathbf{M}}_{j-1/2}^{-1} \tag{26}$$

with the required properties, where $\tilde{M}_{j-1/2}=[\tilde{\mathbf{e}}_1,\tilde{\mathbf{e}}_2,\tilde{\mathbf{e}}_3,\tilde{\mathbf{e}}_4]$ and $\tilde{\mathbf{D}}_{j-1/2}=\mathrm{diag}(\tilde{\lambda}_1,\tilde{\lambda}_2,\tilde{\lambda}_3,\tilde{\lambda}_4)$. The choice of wavestrengths in equations (23a)–(24d) is made by initially considering states $\mathbf{w}_L$ and $\mathbf{w}_R$ that are close to some average state $\mathbf{w}$ as follows.

We seek $\alpha_1,\alpha_2,\alpha_3$ and α_4 such that

$$\Delta\mathbf{w}=\sum_{i=1}^{4}\alpha_i\mathbf{e}_i \tag{27a–d}$$

to second order, where $\mathbf{e}_i$ are given in Section 2 and $\mathbf{w}_L$, $\mathbf{w}_R$ are close to some average state $\mathbf{w}$. After some manipulation we find that equations (27a–d) yield the following expressions for α_i:

$$\alpha_{1,2} = \frac{1}{2a^2}\left(\Delta p \pm \rho \frac{a(Y\Delta u - X\Delta v)}{\sqrt{(X^2+Y^2)}}\right), \tag{28a, b}$$

$$\alpha_3 = \Delta\rho - \Delta p/a^2, \tag{28c}$$

$$\alpha_4 = \rho \frac{X\Delta u + Y\Delta v}{X^2+Y^2}, \tag{28d}$$

where we have made the assumption that to second order

$$\Delta(\rho Z) = Z\Delta\rho + \rho\Delta Z, \quad Z = u, v, H \text{ or } i, \tag{29a–d}$$

$$\Delta(\rho Z^2) = Z^2\Delta\rho + 2\rho Z\Delta Z, \quad Z = u \text{ or } v, \tag{30a, b}$$

and

$$\Delta p = p_\rho \Delta\rho + p_i \Delta i. \tag{31}$$

With the expressions given by equations (28a–d) it is possible to show that

$$\Delta\mathbf{F} = \sum_{i=1}^{4} \lambda_i \alpha_i \mathbf{e}_i \tag{32}$$

to second order.

We now return to the general case, i.e. consider two states $\mathbf{w}_L$, $\mathbf{w}_R$ not necessarily close such that equations (23a)–(24d) are satisfied exactly, where

$$\tilde{\lambda}_i = \tilde{U} \pm \tilde{a}\sqrt{(X^2+Y^2)}, \tilde{U}, \tilde{U}, \tag{33a–d}$$

$$\tilde{\mathbf{e}}_{1,2} = \left(1, \tilde{u} \pm \frac{\tilde{a}Y}{\sqrt{(X^2+Y^2)}}, \tilde{v} \mp \frac{\tilde{a}X}{\sqrt{(X^2+Y^2)}}, \frac{\tilde{p}}{\tilde{\rho}} + \tilde{i} + \tfrac{1}{2}\tilde{u}^2 + \tfrac{1}{2}\tilde{v}^2 \pm \frac{\tilde{a}\tilde{U}}{\sqrt{(X^2+Y^2)}}\right)^{\mathrm{T}}, \tag{34a, b}$$

$$\tilde{\mathbf{e}}_3 = (1, \tilde{u}, \tilde{v}, \tilde{i} + \tfrac{1}{2}(\tilde{u}^2+\tilde{v}^2) - \tilde{\rho}\tilde{p}_\rho/\tilde{p}_i)^{\mathrm{T}}, \tag{34c}$$

$$\tilde{\mathbf{e}}_4 = (0, X, Y, X\tilde{u} + Y\tilde{v})^{\mathrm{T}}, \tag{34d}$$

$$\tilde{\alpha}_{1,2} = \frac{1}{2\tilde{a}^2}\left(\Delta p \pm \frac{\tilde{\rho}\tilde{a}(Y\Delta u - X\Delta v)}{\sqrt{(X^2+Y^2)}}\right), \tag{35a, b}$$

$$\tilde{\alpha}_3 = \Delta\rho - \Delta p/\tilde{a}^2, \tag{35c}$$

$$\tilde{\alpha}_4 = \tilde{\rho}\frac{X\Delta u + Y\Delta v}{X^2+Y^2}, \tag{35d}$$

$$\tilde{U} = Y\tilde{u} - X\tilde{v} \tag{36}$$

and

$$\tilde{a}^2 = \tilde{p}_\rho + \tilde{p}\tilde{p}_i/\tilde{\rho}^2. \tag{37}$$

Thus we have to determine averages $\tilde{\rho}$, $\tilde{u}$, $\tilde{v}$, $\tilde{p}_i$, $\tilde{p}_\rho$, $\tilde{p}$ and $\tilde{i}$ such that equations (23a)–(24d) are satisfied subject to equations (33a)–(37). This problem has a solution and can be determined in a similar way to that of Glaister[5] for the Cartesian case. The required averages are

$$\tilde{Z} = \frac{\sqrt{\rho_L}Z_L + \sqrt{\rho_R}Z_R}{\sqrt{\rho_L} + \sqrt{\rho_R}}, \quad Z = u, v, i \text{ or } H. \tag{38a–d}$$

$$\tilde{\rho} = \sqrt{(\rho_L \rho_R)} \tag{39}$$

and

$$\tilde{p} = \tilde{\rho}(\tilde{H} - \tilde{i} - \tfrac{1}{2}\tilde{u}^2 - \tfrac{1}{2}\tilde{v}^2). \tag{40}$$

In addition,

$$\Delta p = \tilde{p}_\rho \Delta\rho + \tilde{p}_i \Delta i, \tag{41}$$

and suitable approximations satisfying equation (41) are

$$\tilde{p}_\rho = \frac{1}{2}\left[\frac{p(\rho_R, i_R) - p(\rho_L, i_R)}{\Delta\rho} + \frac{p(\rho_R, i_L) - p(\rho_L, i_L)}{\Delta\rho}\right], \quad \rho_L \neq \rho_R, \tag{42a}$$

$$\tilde{p}_\rho = \tfrac{1}{2}[p_\rho(\rho, i_R) + p_\rho(\rho, i_L)], \quad \rho_L = \rho_R = \rho, \tag{42b}$$

$$\tilde{p}_i = \frac{1}{2}\left[\frac{p(\rho_R, i_R) - p(\rho_R, i_L)}{\Delta i} + \frac{p(\rho_L, i_R) - p(\rho_L, i_L)}{\Delta i}\right], \quad i_L \neq i_R, \tag{43a}$$

$$\tilde{p}_i = \tfrac{1}{2}[p_i(\rho_R, i) + p_i(\rho_L, i)], \quad i_L = i_R = i. \tag{43b}$$

(In practice we would replace the conditions $\Delta\rho = 0$, $\Delta i = 0$ with $|\Delta\rho| \leqslant 10^{-m}$, $|\Delta i| \leqslant 10^{-m}$, where the integer m is machine-dependent.) In the case of an ideal gas, equations (42a)–(43b) yield

$$\tilde{p}_\rho = (\gamma - 1)\tfrac{1}{2}(i_L + i_R), \tag{44a}$$

$$\tilde{p}_i = (\gamma - 1)\tfrac{1}{2}(\rho_L + \rho_R). \tag{44b}$$

The approximate Riemann solver presented here can now be implemented as in equations (21a, b), where the required wavespeeds $\tilde{\lambda}_i$, wavestrengths $\tilde{\alpha}_i$ and associated directions $\tilde{\mathbf{e}}_i$ are given by equations (33a)–(43b). In particular, the required approximate Jacobian is given by

$$\tilde{\mathbf{A}}_{j-1/2} = \begin{bmatrix} 0 & Y & -X & 0 \\ Y\left(\tilde{a}^2 - \frac{\tilde{p}_i}{\tilde{\rho}}(\tilde{H} - \tilde{u}^2 - \tilde{v}^2)\right) - \tilde{u}\tilde{U} & \tilde{U} + Y\tilde{u}\left(1 - \frac{\tilde{p}_i}{\tilde{\rho}}\right) & -X\tilde{u} - Y\tilde{v}\frac{\tilde{p}_i}{\tilde{\rho}} & Y\frac{\tilde{p}_i}{\tilde{\rho}} \\ -X\left(\tilde{a}^2 - \frac{\tilde{p}_i}{\tilde{\rho}}(\tilde{H} - \tilde{u}^2 - \tilde{v}^2)\right) - \tilde{v}\tilde{U} & Y\tilde{v} + X\tilde{u}\frac{\tilde{p}_i}{\tilde{\rho}} & \tilde{U} - X\tilde{v}\left(1 - \frac{\tilde{p}_i}{\tilde{\rho}}\right) & -X\frac{\tilde{p}_i}{\tilde{\rho}} \\ \tilde{U}\left(\tilde{a}^2 - \tilde{H} - \frac{\tilde{p}_i}{\tilde{\rho}}(\tilde{H} - \tilde{u}^2 - \tilde{v}^2)\right) & Y\tilde{H} - \tilde{u}\tilde{U}\frac{\tilde{p}_i}{\tilde{\rho}} & -X\tilde{H} - \tilde{v}\tilde{U}\frac{\tilde{p}_i}{\tilde{\rho}} & \tilde{U}\left(1 + \frac{\tilde{p}_i}{\tilde{\rho}}\right) \end{bmatrix}.$$

In the next section we describe a test problem used to test the algorithm of this section.

4. TEST PROBLEM AND NUMERICAL RESULTS

In this section we describe a standard test problem in two-dimensional gas dynamics and give the numerical results achieved for this problem using the Riemann solver described in Section 3.

The problem is that of uniform flow of 'real air' past a circular cylinder. The equation of state used can be written as

$$p = (\gamma(\rho, i) - 1)\rho i,$$

where the form of $\gamma(\rho, i)$ is determined via curve fits to experimental data.[13] The radius of the cylinder is 0·5 and the initial conditions chosen are $\rho = 1{\cdot}4$, $u = 8{\cdot}0$, $v = 0$ and $p = 1$, corresponding to Mach 8 flow. An O-type computational mesh is used and thus the grid transformation is from

Figure 1. Density contours at $t = 0{\cdot}2$; uniform mesh

Figure 2. Density contours at $t = 0{\cdot}4$; uniform mesh

Figure 3. Density contours at $t = 0{\cdot}6$; uniform mesh

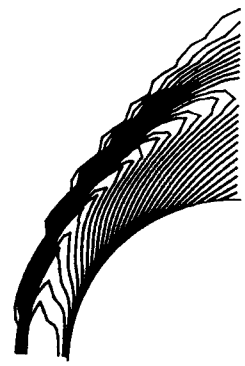

Figure 4. Density contours at $t = 0{\cdot}2$; non-uniform mesh

Figure 5. Density contours at $t = 0{\cdot}4$; non-uniform mesh

Figure 6. Density contours at $t = 0{\cdot}6$; non-uniform mesh

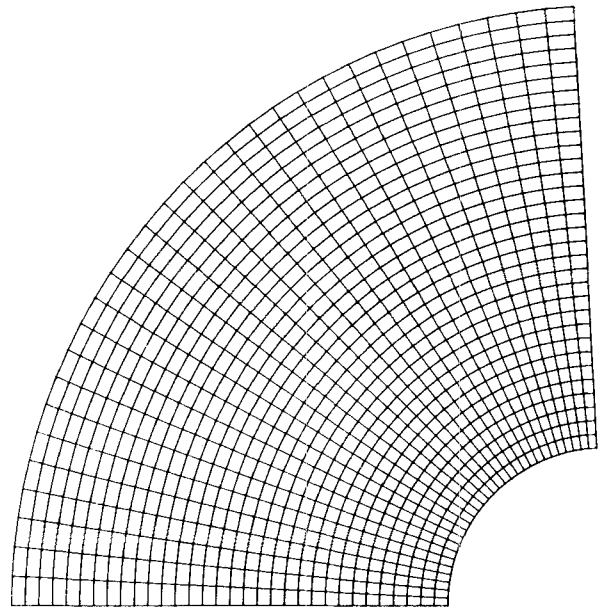

Figure 7. Uniform mesh

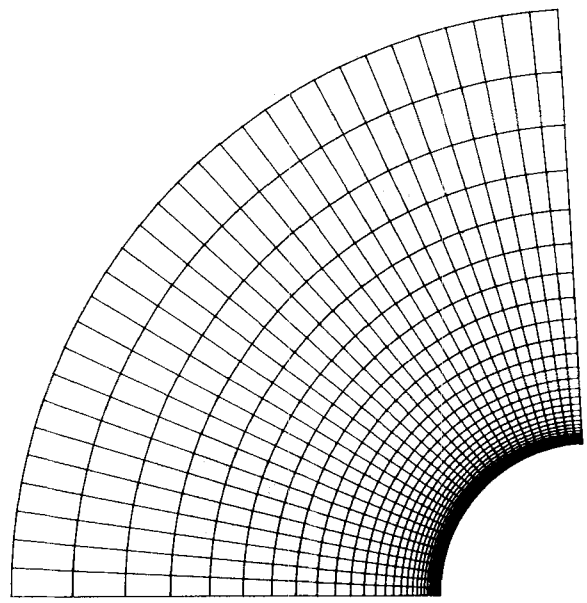

Figure 8. Non-uniform mesh

(x, y) physical space to $(\xi, \eta) \equiv (R, \phi)$ computational space, where R, ϕ are standard plane polar co-ordinates. Because of the line of symmetry along $\phi = \pi$ and the supersonic conditions along $\phi = \pi/2$, the region of computation considered is $(R, \phi) \in [0{\cdot}5, R_{\max}] \times [\pi/2, \pi]$. (The exterior of the boundary is taken as $R_{\max} = 3$ for the computations shown here.) The grid spacing in the ϕ-direction is uniform with 32 grid lines given by $\phi_j = (j - \frac{1}{2})\pi/64 + \pi/2$,

$j = 1, \ldots, 32$. In the R-direction two types of grid spacing are chosen. The first type of grid spacing is uniform with 33 grid lines given by $R_j = (j - \frac{1}{2})5/66 + 0{\cdot}5$, $j = 1, \ldots, 33$. The second grid spacing is of a non-uniform geometric type with 34 grid lines given by $R_1 = 0{\cdot}5 + \frac{1}{2}k$, $R_j = R_{j-1} + k\mu^{j-2}$, $j = 2, \ldots, 34$, where $k = \pi/1280$ and $\mu = 1{\cdot}1648336$. Along $\phi = \pi$ a symmetry boundary condition is applied and along $\phi = \pi/2$ supersonic boundary conditions are applied. Reflecting boundary conditions are applied along the surface of the cylinder $R = 0{\cdot}5$ and inflow conditions are applied along $R = R_{\max}$. The scalar scheme used is first-order;[7] however, a second order TVD (total variation diminishing) scheme could be used.

Figures 1, 2 and 3 display the density contours at $t = 0{\cdot}2$, $0{\cdot}4$ and $0{\cdot}6$ respectively for the mesh with uniform spacing in the R-direction. Corresponding results for the case where the mesh spacing is non-uniform in the R-direction are shown in Figures 4, 5 and 6. In both cases the shock has been captured over at most three cells. Figures 7 and 8 show the uniform mesh and non-uniform mesh respectively. These results are comparable with those of Osher.[14]

5. CONCLUSIONS

We have presented an approximate linearized Riemann solver for two-dimensional compressible flows using body-fitted co-ordinates. The resulting scheme has been applied to supersonic flow of a real gas past a circular cylinder. The numerical results achieved show that the shock has been captured over only a few cells. Furthermore, the scheme developed applies to any convex equation of state and to any regular body-fitted mesh.

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